

Randomness and Solovay degrees

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We consider the behaviour of Schnorr randomness, a randomness notion weaker than Martin-Löf's, for left-r.e. reals under Solovay reducibility. Contrasting with results on Martin-Löf-randomness, we show that Schnorr randomness is not upward closed in the Solovay degrees. Next, some left-r.e. Schnorr random α is the sum of two left-r.e. reals that are far from random. We also show that the left-r.e. reals of effective dimension $> r$, for some rational r , form a filter in the Solovay degrees.

[03D32,68Q30](#)

1 Introduction

Our central notion is Solovay reducibility \leq_S . Informally, for left-r.e. reals α, β , $\alpha \leq_S \beta$ expresses that α is no harder to approximate from the left than β . This implies both $\alpha \leq_T \beta$, and that β is in a particular sense no less random than α . Background is provided e.g. in [5, Chapter 9], and also in [11, Section 3.2].

Randomness notions

The algorithmic theory of randomness defines randomness notions of reals in $[0, 1]$, or equivalently infinite bit sequences, and studies their properties and interactions with computational complexity. The notion of Martin-Löf (ML-)randomness was for a long time considered to be the main one. We review the definition. An open set $U \subseteq [0, 1]$ is r.e. if it is a union of computable sequence of open intervals with rational endpoints, that is, $U = \bigcup_{i \in \mathbb{N}} (p_i, q_i)$ where $\langle p_i \rangle$ and $\langle q_i \rangle$ are computable sequence of rationals. A ML-test is the sequence $\langle U_n \rangle$ of uniformly r.e. open sets with $\mu(U_n) \leq 2^{-n}$ where μ is the Lebesgue measure. A real x is ML-random if $x \notin \bigcap_n U_n$ for every ML-test $\langle U_n \rangle$. A suite of alternative notions has been introduced by modifying this

definition, both stronger and weaker notions than ML-randomness. See e.g. [11, Ch. 3]. Many of these notions have shown their importance in particular for the interaction of randomness and analysis [12, 2].

The following weakening of ML-randomness is important in the present paper. We say that a real x is Schnorr random if it passes all ML-tests $\langle U_n \rangle$ such that $\mu(U_n)$ is computable uniformly in n .

Main results

For left-r.e. reals, ML-randomness interacts closely with Solovay reducibility: a left-r.e. real is ML-random iff it is Solovay complete (Calude et al. [3] and Kučera-Slaman [8]). Also, the complete Solovay degree is join irreducible.

The supremum of two left-r.e. reals in the degree structure induced by Solovay reducibility is given by their arithmetic sum. We are guided by the following two facts that restate some of the results above in terms of the sum. Let α, β be left-r.e. reals.

- (1) If α is ML-random, then $\alpha + \beta$ is ML-random.
- (2) If $\alpha + \beta$ is ML-random, then at least one of α and β is ML-random.

A simple direct proof of the first fact can be found in [11, Theorem 3.2.27]. For the second, see Demuth [4], and as a more recent (and more readable) reference Downey, Hirschfeldt, and Nies [6].

Our first goal is to show that both statements fail for Schnorr randomness (Corollary 3.2 and Theorem 4.1). Thereafter, we prove that in contrast, left-r.e. weakly s -random reals behave similar to ML-random reals (Theorem 5.6). The reals that have effective packing dimension at most s behave similarly to non-ML-random reals (Proposition 5.7).

Preliminaries

Our notation in the algorithmic theory of randomness is standard as in Nies [11] or Downey and Hirschfeldt [5] (except that we write “r.e.” instead of “c.e.”). Unless otherwise stated, reals in this paper are in $[0, 1]$. A real α is called left-r.e. if there exists a computable non-decreasing sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ of rationals such that $\lim_n a_n = \alpha$. We sometimes identify a real α with its binary expansion $X \in 2^\omega$. By $X \upharpoonright n$ we denote the initial segment of X of length n .

2 Solovay reducibility

2.1 Definitions and basic facts

Definition 2.1 (Solovay [13]) For left-r.e. reals α and β , one says that α is Solovay reducible to β , denoted by $\alpha \leq_S \beta$, if there are a constant c and a partial computable function $f : \mathbb{Q} \rightarrow \mathbb{Q}$ such that if $q \in \mathbb{Q}$ and $q < \beta$, then $f(q) \downarrow < \alpha$ and $\alpha - f(q) < c(\beta - q)$.

There is a useful algebraic characterization of Solovay reducibility: $\alpha \leq_S \beta$ means that β can be obtained from α by literally “adding” information, in the sense of adding a left-r.e. real γ . To make this work we also need to scale β . (For the precise version of the result used here see [11, 3.2.29].)

Proposition 2.2 (Downey, Hirschfeldt, and Nies [6]) For left-r.e. reals α, β we have $\alpha \leq_S \beta$ if and only if there are $d \in \mathbb{N}$ and a left-r.e. real γ such that $\alpha + \gamma = 2^d \beta$.

2.2 Connections to K -reducibility

For any reals α, β one writes

$$\alpha \leq_K \beta \text{ if } \forall n K(\alpha \upharpoonright n) \leq^+ K(\beta \upharpoonright n),$$

where K denotes prefix-free Kolmogorov complexity. In view of the Levin-Schnorr theorem, this says that β is no less random than α .

For left-r.e. reals α, β , if $\alpha \leq_S \beta$ then $\alpha \leq_K \beta$ [13]. This fact also follows from Theorem 2.2. We will now obtain an implication in the converse direction under a hypothesis stronger than $\alpha \leq_K \beta$. We write $\alpha \ll_K \beta$ if $\lim_n (K(\beta \upharpoonright n) - K(\alpha \upharpoonright n)) = \infty$.

Proposition 2.3 Let α, β be left-r.e. reals. If $\alpha \ll_K \beta$ then $\alpha <_S \beta$.

In the proposition we can replace $K(\sigma)$ by $C(\sigma)$, $K(\sigma|n)$, or $C(\sigma|n)$ where $n = |\sigma|$. The proof remains almost the same.

Proof First note that $\beta \leq_S \alpha$ would imply $\beta \leq_K \alpha$, contrary to $\alpha \ll_K \beta$. Hence it suffices to show $\alpha \leq_S \beta$.

Let $\langle \alpha_s \rangle_{s \in \mathbb{N}}, \langle \beta_s \rangle_{s \in \mathbb{N}}$ be increasing computable sequences of rationals converging to α, β respectively. Given $n \in \mathbb{N}$, let s_n be the first stage s such that $\alpha_s \upharpoonright n = \alpha \upharpoonright n$. There exists a constant $c \in \mathbb{N}$ such that for each n

$$K(\beta_{s_n} \upharpoonright n) \leq K(\alpha_{s_n} \upharpoonright n) + c = K(\alpha \upharpoonright n) + c.$$

Since $\lim_n (K(\beta \upharpoonright n) - K(\alpha \upharpoonright n)) = \infty$, we have $K(\beta \upharpoonright n) > K(\alpha \upharpoonright n) + c$ for sufficiently large n . Thus, $\beta_{s_n} \upharpoonright n \neq \beta \upharpoonright n$ for all sufficiently large n . Furthermore, $\beta \upharpoonright n$ is not equal to the lexicographic successor of $\beta_{s_n} \upharpoonright n$, for otherwise $K(\beta \upharpoonright n)$ would be within a constant of $K(\beta_{s_n} \upharpoonright n)$. Hence, $\beta - \beta_{s_n} \geq 2^{-n}$.

Define a partial computable function $f : \mathbb{Q} \rightarrow \mathbb{Q}$ by $f(q) = \alpha_s$, where s is the first stage such that $q < \beta_s$. If n is sufficiently large and $s_n \leq s \leq s_{n+1}$, then

$$\alpha - \alpha_s \leq \alpha - \alpha_{s_n} \leq 2^{-n},$$

while

$$\beta - \beta_s \geq \beta - \beta_{s_{n+1}} \geq 2^{-n-1}.$$

Then there is a rational $q^* < \beta$ such that $\alpha - f(q) \leq 2(\beta - q)$ for every rational q with $q^* \leq q < \beta$. Modifying f on the rationals $\leq q^*$ shows that $\alpha \leq_S \beta$.

□

2.3 Connections to ML-randomness

We restate the two guiding facts from the introduction. They connect Solovay reducibility and ML-randomness.

Proposition 2.4 Let α, β be left-r.e. reals. If α is ML-random, then $\alpha + \beta$ is ML-random.

For a direct proof see [11, Theorem 3.2.27]. Alternatively, recall that a left-r.e. real is ML-random iff it is Solovay complete (Calude et al. [3] and Kučera-Slaman [8]). Prop. 2.4 now follows via Prop. 2.2.

The second fact says that the top degree in the Solovay degrees is join irreducible.

Theorem 2.5 (Demuth [4], Downey, Hirschfeldt, and Nies [6]) Let α, β be left-r.e. reals. If $\alpha + \beta$ is ML-random, then at least one of α and β is ML-random.

3 Schnorr randomness is not upward closed in the Solovay degrees

The following will be used to show that for Schnorr randomness, the counterpart of Proposition 2.4 fails.

Theorem 3.1 Let α be a left-r.e. real such that $\forall n K(\alpha \upharpoonright n) < n - f(n)$ for some order function f . There exists a left-r.e. real β such that $\alpha + \beta$ is disjoint from some infinite computable set.

Proof Suppose g, h are strictly increasing computable functions such that the range of h is the complement of the range of g . We define $\gamma = \alpha \oplus_g \beta$ by $\gamma(g(n)) = \beta(n)$ and $\gamma(h(n)) = \alpha(n)$.

If g is a sufficiently fast growing computable function, then $\gamma = \Omega \oplus_g \emptyset$ satisfies $K(\gamma \upharpoonright n) \geq^+ n - \frac{f(n)}{2}$. Note that γ is left-r.e. Since $\alpha \ll_K \gamma$, by Proposition 2.3 we have $\alpha \leq_S \gamma$. By Prop. 2.2 there exist a natural number d and a left-r.e. real β such that $2^d \gamma = \alpha + \beta$. Since γ is disjoint from an infinite computable set, so is $\alpha + \beta$. \square

Partial computable randomness is a notion in between Schnorr and ML-randomness. For the definition see [11, Section 7.4].

Corollary 3.2 The left-r.e. Schnorr random reals are not upward closed in the Solovay degrees. In fact, there is a partial computably random left-r.e. real Solovay below a left-r.e. real that is disjoint from an infinite computable set, and hence not Kurtz random.

Proof By [11, Remark 7.4.17] there exists a left-r.e. partial computably random real α such that that $K(\alpha \upharpoonright n) = O(\log n)$. Now take $\alpha + \beta \geq_S \alpha$ as above. \square

4 Left-r.e. Schnorr random reals can be split into non-Schnorr random reals

We show that the counterpart of Theorem 2.5 for Schnorr randomness does not hold either. We are given a left-c.e. α with some what slow growing initial segment complexity, but α can still be Schnorr random using [11, Remark

7.4.17] as above. We show that α can be written as a sum of left-r.e. reals which are far from random in various sense. For one sense, the effective Hausdorff dimension is 0. We recall the definition here. By [9], for a real $X \in 2^\omega$, the effective Hausdorff dimension $\dim(X)$ can be characterized by

$$(1) \quad \dim(X) = \liminf_n \frac{K(X \upharpoonright n)}{n}.$$

By [1] the effective packing dimension $\text{Dim}(X)$ can be characterized by

$$(2) \quad \text{Dim}(X) = \limsup_n \frac{K(X \upharpoonright n)}{n}.$$

Let \mathbf{a} be a Solovay degree. For $\alpha, \beta \in \mathbf{a}$ we have $\alpha \equiv_K \beta$ and hence $\dim(\alpha) = \dim(\beta)$. So we may well-define $\dim(\mathbf{a}) = \dim(\alpha)$ for some real $\alpha \in \mathbf{a}$. A survey of effective Hausdorff dimension is given in [10].

Theorem 4.1 Let α be a left-r.e. real such that $C(\alpha \upharpoonright n) \leq n - g(n)$ for all n , where g is a computable function such that $\sum_n 2^{-g(n)}$ is finite and a computable real. There exist left-r.e. reals β, γ such that $\alpha = \beta + \gamma$, $\dim(\beta) = \dim(\gamma) = 0$ and both β, γ are not normal and disjoint from infinite computable sets.

Proof Let $\langle \alpha_s \rangle$ be an increasing computable sequence of dyadic rationals converging to α . We impose additional properties to $\langle \alpha_s \rangle$. For each $k \in \mathbb{N}$, we can compute s such that $C_s(\alpha_s \upharpoonright n) < n - g(n)$ for all $n \leq k$. Since α_s is a dyadic rational, $\alpha_s \cdot 2^j$ is a natural number for some j . By taking a subsequence of such s and considering an increasing computable function h that maps k to j , we can assume that $\alpha_s \cdot 2^{h(s)}$ is a natural number and $C(\alpha_s \upharpoonright n) < n - g(n)$ for all s and all $n \leq h(s)$.

Let a_n be the number of s such that the binary representation of $\alpha_{s+1} - \alpha_s$ has a bit 1 at position n . The numbers $\{a_n\}$ are uniformly computably approximable from below and

$$\alpha = \sum_n a_n \cdot 2^{-n}.$$

Notice that $a_n \leq 2^{n-g(n)}$ for each n . For s such that $h(s) < n$, $\alpha_s - \alpha_{s-1}$ does not have a bit 1 at position n . For s such that $h(s) \geq n$, we have $C(\alpha_s \upharpoonright n) < n - g(n)$ and there are at most $2^{n-g(n)}$ many different strings $\alpha_s \upharpoonright n$.

Now we split the natural numbers into intervals I_0, I_1, \dots such that $p(\max(I_n)) > (\min(I_n))^2$ for every n where $p : \mathbb{N} \rightarrow \mathbb{N}$ is a computable function such that

$\sum_{m \geq n} 2^{-g(m)} \leq 2^{-p(n)}$. For these intervals, the following inequalities hold:

$$\sum_{m > \max(I_n)} a_m \cdot 2^{-m} \leq \sum_{m > \max(I_n)} 2^{-g(m)} \leq 2^{-p(\max(I_n))} \leq 2^{-(\min(I_n))^2}.$$

We define two left-r.e. reals β and γ by

$$\beta = \sum_m \{a_m \cdot 2^{-m} : \exists n(m \in I_{2n})\}, \quad \gamma = \sum_m \{a_m \cdot 2^{-m} : \exists n(m \in I_{2n+1})\}.$$

For β , the bits at the positions from $\min(I_n)$ to $(\min(I_n))^2$ are 0 whenever n is odd. This is because, for every n , the terms $a_m \cdot 2^{-m}$ with $m < \min(I_n)$ affect only digits with positions below $\min(I_n)$ and the terms $a_m \cdot 2^{-m}$ with $m > \max(I_n)$ only contribute to a number bounded by $2^{-h(\max(I_n))}$ which has non-zero digits only at positions larger than $(\min(I_n))^2$. By a similar reason, for γ , the bits at the positions from $\min(I_n)$ to $(\min(I_n))^2$ are 0 whenever n is even.

This fact implies $\dim(\beta) = \dim(\gamma) = 0$. To compute $(\min(I_n))^2$ bits of β for odd n , we only need the information of $\min(I_n)$ bits of β and of n . Thus,

$$\frac{K(\beta \upharpoonright (\min(I_n))^2)}{(\min(I_n))^2} \leq \frac{K(\beta \upharpoonright \min(I_n)) + K(n) + O(1)}{(\min(I_n))^2} \rightarrow 0.$$

Furthermore, β and γ are not normal, as the limit superior of the frequencies of subwords of the form 0^k goes to 1 for each k . Finally, β is disjoint from the infinite computable set $\{\min(I_{2n+1}) : n \in \mathbb{N}\}$, and γ from $\{\min(I_{2n}) : n \in \mathbb{N}\}$. \square

5 Filters and Ideals related to effective dimension

In this final section, we study the relationship between effective Hausdorff dimension and Solovay degrees. Recall that a nonempty subset F of a partially ordered set (P, \leq) is a filter if

- (i) for every $x \in F$, $x \leq y$ implies $y \in F$,
- (ii) for every $x, y \in F$, there exists $z \in F$ such that $z \leq x$ and $z \leq y$.

A filter is called principal if $F = \{x : x \geq y\}$ for some $y \in P$.

The ML-random left-r.e. reals are Solovay complete, and therefore induce the trivial filter consisting of the largest element. To treat filters related to dimension, we review the following.

Theorem 5.1 For each rational $r \in (0, 1)$, the set of Solovay degrees \mathbf{a} such that $\dim(\mathbf{a}) > r$ is a filter in the Solovay degrees.

For a proof, we use a generalization of Chaitin's Omega. Let $s \in (0, 1]$. Tadaki [14, Theorems 3.1, 3.2] defined

$$\Omega^s = \sum_{U(\sigma) \downarrow} 2^{-\frac{|\sigma|}{s}}$$

and showed that $s = \dim(\Omega^s) = \text{Dim}(\Omega^s)$. The former equality was independently shown by Mayordomo [9, Corollary 3.3].

Lemma 5.2 Let $r \in (0, 1)$ be a real. Suppose α, β are left-r.e. reals such that $\dim(\alpha) > r$ and $\dim(\beta) > r$. There is a left-r.e. real γ such that $\gamma \leq_S \alpha$, $\gamma \leq_S \beta$ and $\dim(\gamma) > r$.

Proof Let p, q be rationals such that $r < p < q$ and $q < \dim(\alpha), \dim(\beta)$. Then, Ω^p is a left-r.e. real and $\dim(\Omega^p) > r$. Furthermore, by Tadaki's results and (1.2) we have $\Omega^p \ll_K \alpha$, and hence $\Omega^p <_S \alpha$ by Proposition 2.3. Similarly, $\Omega^p <_S \beta$. \square

Theorem 5.1 is now immediate from the lemma.

Question 5.3 Let $r \in (0, 1]$ be a real. If α, β are left-r.e. reals such that $\dim(\alpha) \geq r$ and $\dim(\beta) \geq r$, does there exist $\gamma \leq_S \alpha, \beta$ with $\dim(\gamma) \geq r$?

Equivalent, is $\{\mathbf{a} : \dim(\mathbf{a}) \geq r\}$ a filter? By the following it is not a principal filter.

Proposition 5.4 Let α be a left-r.e. real such that $\dim(\alpha) = r > 0$. There exists a left-r.e. real β such that $\dim(\beta) = r$ and $\beta <_S \alpha$.

Proof Let $f(n)$ be the biggest natural number less than $n + \sqrt{n}$. Note that $f : \mathbb{N} \rightarrow \mathbb{N}$ is an increasing computable function. Given a left-r.e. α , let $\beta = \{f(n) : n \in \alpha\}$. Notice that β is a left-r.e. real with the approximation $\beta_s = \{f(n) : n \in \alpha_s\}$ where α_s is an increasing computable approximation of α .

Now the following equalities hold:

$$\begin{aligned} r &= \liminf \frac{C(\alpha \upharpoonright n)}{n} = \liminf \frac{C(\beta \upharpoonright f(n))}{n} \\ &= \liminf \frac{C(\beta \upharpoonright f(n))}{f(n)} \cdot \frac{f(n)}{n} = \liminf \frac{C(\beta \upharpoonright n)}{n}. \end{aligned}$$

To describe $\beta \upharpoonright f(n)$ one only needs to describe the bits in the positions $f(k)$ for $k < n$. Since $f(n+1) - f(n) \leq 2$ for every n , $C(\beta \upharpoonright n) = C(\beta \upharpoonright f(k)) + O(1)$ for some k .

Clearly $\beta \leq_S \alpha$. Assume by way of contradiction that the converse also holds. Then, there are constants $c, c' \in \mathbb{N}$ such that

$$C(\alpha \upharpoonright f(n)) \leq C(\beta \upharpoonright f(n)) + c' \leq C(\alpha \upharpoonright n) + c.$$

Let $k \in \mathbb{N}$. Then, there exists $a_0 \in \mathbb{N}$ such that $f(n) - n \geq k$ for all $n \geq a_0$. Inductively, we define $a_{n+1} = f(a_n)$. Then, $a_n \geq a_0 + kn$. In contrast,

$$C(\alpha \upharpoonright a_n) \leq C(\alpha \upharpoonright a_{n-1}) + c \leq C(\alpha \upharpoonright a_0) + cn.$$

Hence,

$$\liminf \frac{C(\alpha \upharpoonright a_n)}{a_n} \leq \liminf \frac{C(\alpha \upharpoonright a_0) + cn}{a_0 + kn} \leq \frac{c}{k}.$$

Since k is arbitrary, we have $\dim(\alpha) = 0$, which is a contradiction. \square

Next we consider partial randomness.

Definition 5.5 (Tadaki [14], see also [5, Definition 13.5.1]) A test for weak s -ML randomness is a sequence of uniformly r.e. sets of strings $\langle V_k \rangle$ such that $\sum_{\sigma \in V_k} 2^{-s|\sigma|} \leq 2^{-k}$. A set $X \in 2^\omega$ is weakly s -ML-random if $X \notin \bigcap_k \llbracket V_k \rrbracket$ for all tests for weak s -ML randomness.

The strings can be replaced with open intervals with dyadic rational endpoints [5, after Definition 13.5.8].

Tadaki [14] showed that X is weakly s -ML-random if and only if $K(X \upharpoonright n) > sn - O(1)$. Thus for left-r.e. reals, being weakly s -ML-random is upward closed under \leq_S .

Theorem 5.6 Let $s \in (0, 1] \cap \mathbb{Q}$. The set of left-r.e. weakly- s -random Solovay degrees is a principal filter in Solovay degrees with the degree of Ω^s as the bottom element.

Proof We follow the argument of Kučera-Slaman theorem [8]. Let α be a left-r.e. weakly s -random real. The goal is to show $\Omega^s \leq_S \alpha$.

We construct a test for weak s -ML-randomness $\langle U_k \rangle_{k \in \mathbb{N}}$. At stage t act as follows. If $\alpha_t \in U_k[t]$, then do nothing. Otherwise let t' be the last stage at

which we put anything into U_k (or $t = 0$ if there is no such stage), and put the intervals $(\alpha_t, \alpha_t + 2^{-\frac{k}{s}}(\Omega_t^s - \Omega_{t'}^s))$ into U_k .

Let $t_0 = 0, t_1, \dots$ be stages we put anything into U_k . Then, the total weight is bounded by

$$\sum_i (2^{-\frac{k}{s}}(\Omega_{t_{i+1}}^s - \Omega_{t_i}^s))^s \leq \sum_{U(\sigma)\downarrow} (2^{-\frac{k}{s}})^s (2^{-\frac{|\sigma|}{s}})^s \leq 2^{-k}.$$

Thus, $\langle U_k \rangle$ is a test for weak s -ML-randomness.

Since α is weakly s -random, there exists a constant k such that $\alpha \notin U_k$. Then,

$$\alpha_{t_{i+1}} - \alpha_{t_i} > 2^{-k/s}(\Omega_{t_i}^s - \Omega_{t_{i-1}}^s)$$

for each $i > 0$, so $\Omega^s \leq_S \alpha$. □

Now we turn to ideals, the dual notion of filters. A nonempty subset I of a partially ordered set (P, \leq) is an ideal if

- (i) for every $x \in I, y \leq x$ implies $x \in I$,
- (ii) for every $x, y \in I$, there exists $z \in I$ such that $x \leq z$ and $y \leq z$.

Theorem 2.5 says that the non-ML-random Solovay degrees form an ideal. The following is an easy observation.

Proposition 5.7 For each $r \in [0, 1]$, the set of left-r.e. degrees \mathbf{a} such that $\text{Dim}(\mathbf{a}) < r$ is an ideal of left-r.e. Solovay degrees. We can replace $\text{Dim}(\mathbf{a}) \leq r$ with $\text{Dim}(\mathbf{a}) < r$.

Proof Let α, β be left-r.e. reals such that $\text{Dim}(\alpha) < r$ and $\text{Dim}(\beta) < r$. Then, $\gamma = \alpha + \beta$ satisfies $\alpha \leq_S \gamma$ and $\beta \leq_S \gamma$. One can easily check that $K(\gamma \upharpoonright n) = \max(K(\alpha \upharpoonright n), K(\beta \upharpoonright n))$ ([7, Theorem 7.4]). Hence,

$$\text{Dim}(\gamma) = \limsup \frac{K(\gamma \upharpoonright n)}{n} \leq \limsup \frac{\max(K(\alpha \upharpoonright n), K(\beta \upharpoonright n))}{n} < r.$$

The proof of the case of $\text{Dim}(\mathbf{a}) \leq r$ is almost the same. □

The set of left-r.e. K -trivial degrees is an ideal in the Solovay degrees because the sum of two K -trivial reals is again K -trivial [7, Theorem 7.2].

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